Green Level

Mathcat 2020

1. **(5 credits)** Potter and Malfoy travel in neighbouring coaches of the Hogwarts Express. Potter is in the fifth coach from the front of the Express, and Malfoy is in the seventh coach from the rear of the Express. How many coaches does the Express have? Find all possible answers.

   Answer: 10 or 12 coaches
   Solution: Let the coaches be numbered from the front of the Express; this means that Potter is in the coach number 5. Since Malfoy is in a neighbouring coach, the number of his coach is either 4, or 6. Therefore, the Express has either $4+7-1=10$, or $6+7-1=12$ coaches.

2. **(5 credits)** At the mutant school, all students sit in twos at a table. 60 percent of boys have a boy as a table mate, and 20 percent of girls have a girl as a table mate. How many times greater is the number of boys in this school than the number of girls?

   Answer: 2 times
   Solution: According to the problem statement, 40% of boys have a girl as a table mate, and 80% of girls have a boy as a table mate. Therefore, 40% of boys and 80% of girls is the same number equal to the number of tables with a boy and a girl as table mates. This means that the number of boys in this school is 2 times greater than the number of girls.

3. **(7 credits)** Captain Jack Sparrow, Elizabeth Swann and Will Turner divide the treasure of Davy Jones among themselves. First Jack gave each of his friends one quarter of his coins and half a coin. Then Elizabeth gave each of her friends one quarter of her coins and half a coin. Then Will did the same. As a result, everyone got 30 coins. How many coins did each of them have originally?

   Answer: Jack had 14 coins, Elizabeth had 26 coins and Will had 50 coins
   Solution: Each pirate gave one half of his or her coins and a coin to his or her friends at every step. This means that Will had $(30+1)\cdot 2=62$ coins, and both Jack and Elizabeth had $30-16=14$ coins before the last step. Elizabeth had $(14+1)\cdot 2=30$ coins, Jack had $14-8=6$ coins, and Will had $62-8=54$ coins before the next to last step. Therefore, Jack originally had $(6+1)\cdot 2=14$ coins, Elizabeth had $30-4=26$ coins, and Will had $54-4=50$ coins.

4. **(8 credits)** 20 students of the Police Academy play 'Civilians and Terrorists'. In the first round, each of them got a badge 'a civilian' or 'a terrorist'. Civilians always speak the truth and terrorists always lie. In the second round, the students got the same set of 20 badges, upon which 6 students told that their roles changed and other 14 students told that they have the same roles as in the first round. What is the possible number of the badges marked 'a civilian'? Find all possible answers.

   Answer: 14 badges
   Solution: A student having the badge 'a civilian' has to tell in the second round that he or she has the same role. Indeed, if he becomes a civilian in the second round, he tells the truth, and if he becomes a terrorist, he lies. Similarly, a student having the badge 'a terrorist' has to tell in the second round that his or her role changed. Therefore, 14 students had the badge 'a civilian' in the first round, and other players had the badge 'a terrorist'.

5. **(10 credits)** Each face of a cube was split into four identical squares, and then those squares were painted several colours so that all squares having a common side appeared to be painted different colours. What is the largest possible number of squares of the same color?

   Answer: 8 squares
   Solution: One vertex of each square coincides with one corner of the cube. At the same time, three squares adjacent to each corner of the cube are to be painted different colors. Therefore, if the number of squares is more than eight, they cannot be painted the same color. Moreover, if one of the squares adjacent to each corner of the cube is painted blue and all the other squares are painted colors other than blue, then the cube has exactly eight blue squares, and this satisfies the problem statement.

6. **(10 credits)** Three hackers want to crack a website. The second and the third hackers can do it together 2 times faster than the first one, and the first and the third hackers can do it together 3 times faster than the second one. How many times faster can the first and the second hackers crack the website than the third one?

Answer: 1.4 times
Solution: Let the performance of the first, second and third hackers be x, y and z respectively. In this case, the problem statement can be shown as the system of two following equations: $y+z=2x$ and $x+z=3y$. We can solve this system for such x and y, but another method can also be used: we multiply the first equation by 4 and the second one by 3 and then add the results. From the equation $3x+4y+7z=8x+9y$ we get $x+y=1.4z$, and this means that the first and second hackers can crack the website 1.4 times faster than the third one.

7. **(12 credits)** Several knights sit at the round table. Each of them has some daggers and some arrows. If two knights are able to equally share their daggers and arrows (each kind of weapon is to be shared separately), they sit next to each other. What is the maximum number of the knights sitting at the table?

Answer: 8 knights
Solution: For each knight, we write down a pair of numbers: the remainders of dividing the numbers of his daggers and arrows by 2. Two knights can equally share their daggers and arrows only if such pairs of numbers are the same. Now it is clear that the same pair of numbers can be written down for two knights at most, since there are always two knights not sitting next to each other out of any three ones (except for the case when there are only three knights, but we are not interested in this case, since we are looking for the maximum number of the knights). And since there can be only four different pairs, namely (0, 0), (0, 1), (1, 0), (1, 1), there are no more than eight knights at the table. If each of these four pairs can be found twice and the knights with the same pairs of numbers sit next to each other, then all conditions are met.

8. **(13 credits)** A boy cut out a square with a side of 12 cm and a triangle from a page of his father's passport. He managed to cover at least three quarters of the area of the triangle with the square, but only half of the square could be covered with the triangle. What is the area of the triangle?

Answer: 96
Solution: We arrange the square so that it covers the largest possible part of the area of the triangle. This arrangement gives us the maximum possible overlapping area of the figures, therefore the triangle also covers the largest possible part of the area of the square. This means that three quarters of the area of the triangle is equal to half the area of the square, which is $12\cdot12:2=72$, therefore, the area of the triangle is $72\cdot4:3=96$.

9. **(15 credits)** 100 students took part in the contest where four problems were posed to them. The first problem was solved by 90 contestants, the second one was solved by 80 contestants, the third one was solved by 70 contestants, and the fourth one was solved by 60 contestants. Nobody solved all four problems. All contestants who solved the third and the fourth problems were announced as winners. How many winners were announced?

Answer: 30 winners
Solution: We can distinguish four groups of contestants: in the first group we include 10 students who did not solve the first problem, in the second one we include 20 students who did not solve the second problem, in the third one we include 30 students who did not solve the third problem, and in the fourth one we include 40 students who did not solve the fourth problem. Since nobody solved all four problems, each contestant is included into at least one group. At the same time, the total number of students in the groups is $10+20+30+40=100$, and this means that these groups do not intersect. Therefore, there are exactly $30+40=70$ contestants who did not solve the third or fourth problem, and the remaining $100-70=30$ contestants solved both these problems and were announced winners.

10. (15 credits) What is the largest number of integers that can be arranged in a row so that the sum of any 49 consecutive numbers is even, and the sum of any 50 consecutive numbers is odd?

Answer: 97 integers

Solution: According to the problem statement, the first and last numbers in any group of 50 consecutive numbers are odd. If we had at least 98 numbers, then a group of 50 consecutive numbers could start from the first number in the row, or from the second one, ... or from the 49th one in the row. But in such case all these 49 numbers would be odd, and their sum would also be odd, but this contradicts the problem statement. This means that no more than 97 numbers can be used. If we arrange 48 ones, zero and another 48 ones in a row, then each group of 49 or 50 consecutive numbers contains zero in 49th place, therefore, the sums of numbers in such groups are even and odd respectively.