Yellow Level

1. (5 credits) The perimeter of the square was increased by 5 percent. By how many percent did its area increase?

(Answer: 10.25 percent.)

Solution. Let a be the length of the side of the initial square. After the increase, it became equal to $1.05*a$. This means that the area was increased from a^2 to $(1.05*a)^2 = 1.1025 \times a^2$, i.e. it was increased by 10.25 percent.

2. (5 credits) Each of the six members of the team was solving 9 problems. 2 hours later the members found that each of them solved 2 problems. A problem was considered solved if it was solved by three members. What could be the minimum number of the problems unsolved by the team?

(Answer: 5.)

Solution. Total number of the mentioned problems solved by the members of the team was $6x2=12$, and each problem was sold three times. Therefore, 4 different problems were solved and 5 remaining problems were not solved.

3. (6 credits) A block of sheets fell out of the book having pages with standard pagination. Page 12 was on the first sheet of the block, and page 67 was on its last sheet. How many sheets fell out of the book?

(Answer: 29.)

Solution. Since a book sheet starts with an odd-numbered page and ends with an even-numbered page, the block of sheets that fell out of the book contained pages 11-68. Therefore, total number of such pages was equal to $(68-11+1)/2=29$.

4. (8 credits) Solve the equation system $a^2+2b=-1$; $b^2-2a=-1$.

(Answer: $a=1$, $b=-1$.)

Solution. After adding one equation to the other, we get $a^2+b^2+2b-2a+2=0$, $(a-1)^2+(b+1)^2=0$. Therefore, we can immediately conclude that $a=1$, $b=-1$.

5. (10 credits) Santa Claus put 13 chocolates in each of his Christmas gift boxes, but the remaining number of his chocolates was less than needed for the last gift box. When he began to put 15 chocolates instead of 13 ones in each gift box, all his chocolates happened to be distributed in the same gift boxes. What is the maximum number of the chocolates that Santa Claus could have?

(Answer: 90.)

Solution. If all Santa's chocolates were distributed in n gift boxes, Santa Claus had at least $15*n$ chocolates, but definitely not $13*(n+1)$ chocolates, because in such case, when he put 13 chocolates in each box, the number of gift boxes would be $n+1$. Therefore, $13*n+13>15*n$, and this means that $n<6.5$. Such maximum possible n is 6, therefore the number of the chocolates does not exceed 90. It is easy to see that this number satisfies the conditions of the problem.

6. (12 credits) A computer game player can score 8, 9 or 19 points at each level (depending on his success at that level). What is the maximum natural number of points that cannot be scored in this game?

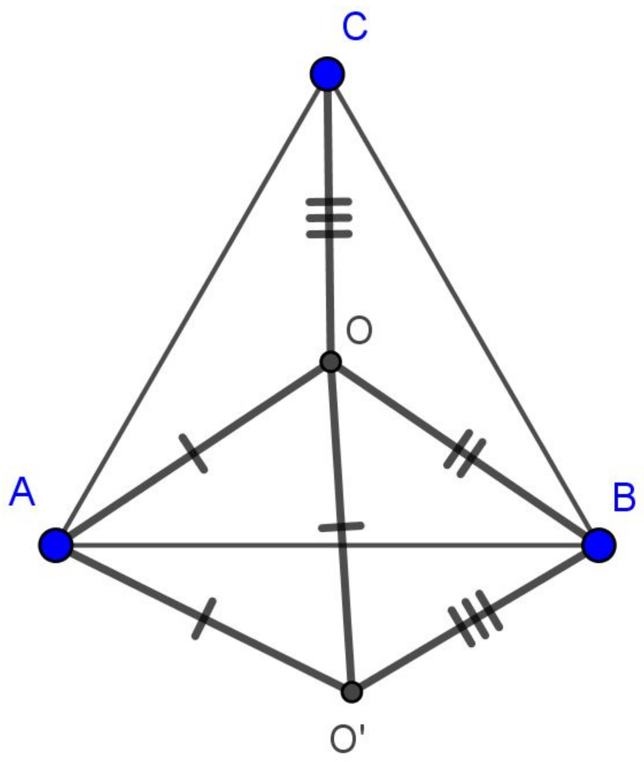
(Answer: 39.)

Solution. Take, for example, the number $n*8$, where n is a natural number. Here are subsequent terms of the sequence: $(n-1)*8+9$, $(n-2)*8+9*2$, $(n-2)*8+19$, $(n-3)*8+19+9$, $(n-4)*8+19+9*2$, $(n-4)*8+19*2$. It is impossible to obtain the eighth term using this approach, so it is obvious that the formula of the answer is $n*8+7(n\geq 4)$. At the same time, when $n=4$, the number 39 cannot be obtained by using several summands equal to 8, 9 or 19, but the number 45 ($n=5$) can be represented in the form 45=9*5. All subsequent terms can be obtained by adding 8 to the previous ones.

7. (12 credits) The point O is inside the equilateral triangle ABC. It is known that ∠AOB=112°, ∠BOC=123° and ∠COA=125°. Find all angles of the triangle having sides equal to AO, BO and CO.

(Answer: 63°, 65°, 52°.)

Solution.



Turn the part of the picture around the vertex A by 60°. In this case, C goes to B, and the point O goes to the new point O'. Therefore, the segment OC goes to O'B. Since the points O, O' and A form a regular triangle, OO' = OA. Thus, the sides of the triangle OBO' are equal to OA, OB and OC. However, it is easy to find the angle O'OB = AOB-60° = 52° from the figure. The other two angles can be found similarly (by using other rotations): 125°-60°=65° and 123°-60°=63°.

8. (12 credits) Petya tries to find grid rectangles, diagonals of which cross exactly 23 squares. He has already found 1×23 and 23×23 rectangles. How many more such rectangles can he find (a×b and b×a rectangles are assumed to be identical)?

(Answer: Three: 5×19, 7×17 or 11×13.)

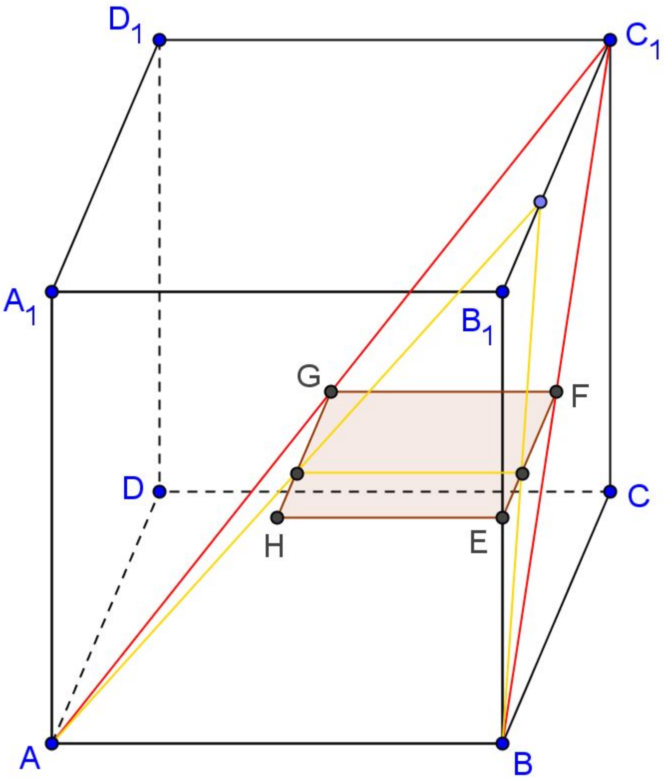
Solution. The diagonal of the a×b rectangle intersects all a-1 horizontal lines and b-1 vertical lines. If the numbers a and b do not have common divisors, all these lines are intersected at different points, and there are (a-1)+(b-1) intersection points on the diagonal, and it is divided into a+b-1 line segments, i.e. the diagonal intersects a+b-1 cells. This gives us the equation a+b-1=23, i.e. a+b=24 (provided that there are no common divisors). To enable a and b (the sum of which is 24) not to have common divisors, each term must be odd and not divisible by 3. This happens for pairs (1,23), (5,19) and (7,17). If a and b have common divisors, then some points of intersection of the diagonal with the horizontals and verticals coincide, and the number of such points is equal to LCD(a,b) (LCD means 'least common divisor'). Reasoning similarly to the main case, we may conclude that this diagonal intersects a+b-LCD(a,b) cells. The equality a+b-LCD(a,b)=23 may be valid for the case LCD(a,b)> 1 only if LCD(a,b)=23, because the left side is divisible by LCD(a,b). For this case we have a=23a', b=23b', a+b=68, a'+ b'=2, therefore a'= b'=1. Thus, the only option a=b=23 already found by Petya may be added here.

9. (15 credits) The length of the edge of the cube ABCDA1B1C1D1 is 8. The edges AB and CC1 are connected by all possible line segments. Midpoints of the segments form a plane figure (see fig.). What is the area of this figure?

(Answer: 16.)

Note: this figure is a square whose side is equal to half the edge of the cube.

Solution. If one end of the segment is in the point C1, and the other end moves along AB, the middle of this segment moves along the midline of the triangle ABC1, i.e. along the segment GF. If C1 moves along the edge B1C1, the midline moves from the position of the segment GF to the segment HE. Thus, the midpoints of the segments fill the square EFGH.



10. (15 credits) Specify the fractions of type 1/A and 1/B (A and B are natural numbers) the difference of which is 4/221.

(Answer: 1/13 and 1/17, 1/51 and 1/663, 1/52 and 1/884, 1/55 and 1/12155.)

Solution.

1/a-1/b=4/221

221(b-a)=4ab

(221-4a)(221+4b)=221^2=13^2*17^2

Thus, 221^2 needs to be decomposed into two factors: the lesser one is 221-4a, and the greater one is 221+4b. The lesser factor may be equal to 1, 13, 17 or 13^2. This gives the following values: 221-4a=1, 221-4a=13, 221-4a=17 and 221-4a=169, i.e. a=55, a=52, a=51, a=13. Accordingly, 221+4b=48841, 221+4b=3757, 221+4b=2873, 221+4b=289, i.e. b=12155, b=884, b=663, b=17.